University of Maryland

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Soil Dynamics & Earthquake Engineering (ENCE743)

April 30, 2022

Technical Report – Stress Wave in Piles

Localization & Quantification of Damages due to Pile Driving

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ACADEMIC YEAR 2021-2022

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1. Introduction

This report will cover the topic of stress wave through piles when hammered into soil. The main purpose is to use wave equations to find out if a pile has been damaged during the hammering process, and if so, how severe the damage is.

This is extremely useful to do while hammering as it could be possible to stop the process if damage starts to appear before the damage is too severe and the pile can no-longer be used. In addition to this, the understanding of the severity of the damage gives a good estimate of how much strength is left, and therefore useful as bearing capacity.

The report will be divided into two parts, the first part will be the derivation and definition of the stresses through piles, using wave equations and the transmission-reflection principles of waves. The second part will be more focused on the practical side and how forces and velocities are measured in the field, and how these (combined with the wave equations) can be used to estimate the damage on a pile.

The research done so far does not seem to give a comprehensive solution to the problem, it is therefore to be expected a certain level of assumptions since no experimentation can be carried out for the report.

2. Wave Theory

In order to understand what happens through a pile, it if first necessary to understand how stresses and forces propagate through the pile. To understand the physical process through mathematical analysis, this report will consider the problem as a one-dimensional system where the pile is a “rod”.

2.1. Compression Wave

When force impacts on a rod, the particles of the rod will compress, and this causes the stress. This compression propagates as a wave throughout the whole rod, and therefore the stresses propagate as well (Abramson, Plass & Ripperger, 1958). The propagation of a compression wave can be described as follows (McNiven & McCoy, 1974):

$$c^2 \frac{\partial^2 u_x}{\partial x^2} = \frac{\partial^2 u_x}{\partial t^2} \qquad (1)$$

Where u_x is the displacement of a particle of the rod in the x-direction and $c = \sqrt{E/\rho}$ is the speed of sound in the material.

2.2. Stress Inside the Rod

This is the simplest form to represent the propagation of compression waves through a rod, but it is extremely useful to derive the stresses at any point of the rod. In order to do this, the simple Newtown’s second law (the law of motion) will be used:

$$F = m \cdot a \qquad (2)$$

Considering $a = \frac{dv}{dt}$ we can state that:

$$F = m \cdot \frac{dv}{dt} \tag{3}$$

Since only a very small section of the rod is studied, it is possible to say that $m = \rho A dx$ which can then be substituted in the formula to get:

$$F = \rho A dx \cdot \frac{dv}{dt} \tag{4}$$

Knowing that $F = \sigma A$ we can state the following:

$$\sigma A = \rho A dx \cdot \frac{dv}{dt} \tag{5}$$

And by re-arranging:

$$\sigma = \rho \frac{dx}{dt} \cdot dv \tag{6}$$

Where $\frac{dx}{dt} = c$ is the sound wave velocity. This equation can also be derived from the impulse equation, which in turns is derived itself from Newton’s law of motion (Svensson & Tell, 2015).

Since the next particle has a velocity of zero before the previous particle collides, $dv = v_{t+1} - v_t$ will simply be v_{t+1}, or the particle velocity itself (v_p) (Svensson & Tell, 2015):

$$\sigma = \rho c v_p \tag{7}$$

With ρc known as the specific impedance of the material.

2.3. Reflection and Transmission of Compressive Waves

When waves encounter any type of boundary, two other waves will be generated: a reflective and a transmissive wave (Schwartz, 2013). To understand the behavior of these waves, the magnitude of them, and how they can impact the model, let us imagine that every particle is a spherical mass.

In this model the particle with mass m_1 is the one on the left side of the boundary which will be the first impacted by the stress (or compressive) wave. The particle with mass m_2 is the one on the right side of the boundary. Let us define v_i as the velocity of m_1 prior to the collision with the boundary, and let us use the principle of conservation of momentum to define the velocity (v_r) of m_1 and the velocity (v_t) of m_2 after collision (Schwartz, 2013):

$$m_1 \cdot v_i = -v_r \cdot m_1 + v_t \cdot m_2 \tag{8}$$

Note that the reflected velocity (v_r) is negative since at impact it is going to be opposite to the impulse velocity and the transmitted velocity.

By manipulating the equation, it is possible to state the following:

$$v_t = \frac{m_1}{m_2} \cdot (v_i + v_r) \tag{9}$$

From the principle of conservation of energy (Schwartz, 2013):

$$\frac{1}{2}m_1 \cdot {v_i}^2 = \frac{1}{2}{v_r}^2 \cdot m_1 + \frac{1}{2}{v_t}^2 \cdot m_2 \quad (10)$$

Combining equation 10 with the previously derived equation 9 it is possible to state that:

$$\frac{1}{2}m_1 \cdot {v_i}^2 = \frac{1}{2}{v_r}^2 \cdot m_1 + \frac{1}{2}\left[\frac{m_1}{m_2} \cdot (v_i + v_r)\right]^2 \cdot m_2 \quad (11)$$

And so:

$$\frac{1}{2}m_1 \cdot {v_i}^2 = \frac{1}{2}{v_r}^2 \cdot m_1 + \frac{1}{2}\left[\frac{m_1}{m_2} \cdot (v_i + v_r)\right]^2 \cdot m_2 \quad (12)$$

With some algebra, and by applying the quadratic formula, it is possible to find that:

$$v_r = \frac{(m_2 - m_1)}{m_2 + m_1} \cdot v_i \quad (13)$$

Using this equation in equation 9:

$$v_t = \frac{m_1}{m_2} \cdot \left(v_i + \frac{(m_2 - m_1)}{m_2 + m_1} \cdot v_i\right) \quad (14)$$

With which it is possible to draw the final conclusions:

$$v_t = \frac{2m_1}{m_2 + m_1} v_i \quad (15)$$

$$v_r = \frac{(m_2 - m_1)}{m_2 + m_1} \cdot v_i \quad (16)$$

The previous formulae work for masses, but for solids (and fluids) the “masses” are actually the specific impedances of the different materials (Schwartz, 2013) leading to:

$$v_t = \frac{2\rho_1 c_1}{\rho_2 c_2 + \rho_1 c_1} v_i \quad (17)$$

$$v_r = \frac{(\rho_2 c_2 - \rho_1 c_1)}{\rho_2 c_2 + \rho_1 c_1} \cdot v_i \quad (18)$$

The derivation of these two equations can also be performed using only mathematics (without momentum and energy equations) by stating the boundary conditions of the problem.

This suggests that if a random section inside the rod is taken to be studied, and so the material before and after the boundary are the same ($\rho_2 = \rho_1$), then $v_t = v_i$ and $v_r = 0$. This is actually what happens at any section of the rod, and an example of this phenomenon can be found in the Newton’s cradle where when one balls hits the next, the previous one remains still. It is therefore possible to conclude that on impact, if the material of the rod is homogeneous, the reflected velocity is zero, and all the velocity is transmitted to the next particle (after the boundary); therefore, using equation 7, if the velocity of the next particle is the same as the pervious one, the stress will be the same as well.

Following the same principles, it is possible to see that if there is a fixed-end boundary, the impedance $\rho_2 c_2$ will simply be extremely high, making the ratio $\frac{(\rho_2 c_2 - \rho_1 c_1)}{\rho_2 c_2 + \rho_1 c_1} \approx 1$; at the same time $\frac{2\rho_1 c_1}{\rho_2 c_2+\rho_1 c_1} \approx 0$, this implies that at a fixed-end boundary, the impulse velocity will be almost entirely reflected. On the contrary, if there is a free-end boundary, the impedance $\rho_2 c_2$ will be close to zero, which would make $\frac{(\rho_2 c_2-\rho_1 c_1)}{\rho_2 c_2+\rho_1 c_1} \approx -1$ and $\frac{2\rho_1 c_1}{\rho_2 c_2+\rho_1 c_1} \approx 2$, implying that at a free-end boundary the transmitted velocity is double the impulse velocity, and that the rod will experience tension (Schwartz, 2013; Svensson & Tell, 2015; Chapman & Ward, 1990).

2.4. Study of the Stress at a Boundary

For this report, the particle velocities at a boundary are not of primary importance, but it was necessary to derive them in order to be able to subsequently derive the stresses, which is the value that is the most important to be able to quantify damages on a pile.

In this section the connection between the stresses and velocities will be made to derive the final equation that models the reflected stress and the transmitted stress using the previously derived equations:

$$\sigma = \rho c v_p \tag{19}$$

$$v_t = \frac{2\rho_1 c_1}{\rho_2 c_2 + \rho_1 c_1} v_i \tag{20}$$

$$v_r = \frac{(\rho_2 c_2 - \rho_1 c_1)}{\rho_2 c_2 + \rho_1 c_1} \cdot v_i \tag{21}$$

By combining these equations, it is possible to state the following:

$$\frac{\sigma_t}{\rho_2 c_2} = \frac{2\rho_1 c_1}{\rho_2 c_2 + \rho_1 c_1} \cdot \frac{\sigma_i}{\rho_1 c_1} \tag{22}$$

$$\frac{\sigma_r}{\rho_1 c_1} = \frac{(\rho_2 c_2 - \rho_1 c_1)}{\rho_2 c_2 + \rho_1 c_1} \cdot \frac{\sigma_i}{\rho_1 c_1} \tag{23}$$

And so:

$$\sigma_t = \frac{2\rho_2 c_2}{\rho_2 c_2 + \rho_1 c_1} \cdot \sigma_i \tag{24}$$

$$\sigma_r = \frac{(\rho_2 c_2 - \rho_1 c_1)}{\rho_2 c_2 + \rho_1 c_1} \cdot \sigma_i \tag{25}$$

With these formulae it is possible to notice that if the boundary should be a fixed-end type, the impedance $\rho_2 c_2$ will tend to infinity, making the transmitted stress double the impulse stress $\sigma_t = 2\sigma_i$; while the reflected stress will be equal to the impulse stress. On the other hand, if the boundary is a free-end type, the impedance $\rho_2 c_2$ will tend to zero, making the transmitted stress equal to zero, while the reflected stress would be equal and opposite (indicating a tensile stress) to the impulse stress (Svensson & Tell, 2015). Since both stresses, on the left side of the boundary, and on the right side of the boundary must be the same, the following equation must be satisfied:

$$\sigma_i + \sigma_r = \sigma_t \tag{26}$$

It is therefore evident that the worst-case scenario is where there is an induced stress in the rod which then encounters a fixed-end boundary (a very dense material) and causes a stress, at the boundary, double the impulse stress. This is the reason why hammering a pile on a rock layer can cause serious damages to the pile, especially at the toe.

3. Assessing Pile Damage with Wave Equations

3.1. From Stress to Force

For the following part of the report, it will be necessary to introduce the cross-sectional area of the pile, and for this reason it is important to convert the formulae from stress to force, so that the area can be easily visualized as a factor playing a great role in the equations themselves:

$$F = A\rho c v_p \tag{27}$$

$$F_t = \frac{2\rho_2 c_2 A_2}{\rho_2 c_2 A_2 + \rho_1 c_1 A_1} \cdot F_i \tag{28}$$

$$F_r = \frac{(\rho_2 c_2 A_2 - \rho_1 c_1 A_1)}{\rho_2 c_2 A_2 + \rho_1 c_1 A_1} \cdot F_i \tag{29}$$

3.2. Localizing a Pile Damage

To assess pile damages, the easiest solution is to measure the force and velocity at the top of the pile. It is therefore clear that the time needed for a reflected force to be measured will be:

$$t_r = \frac{2L}{c} \tag{30}$$

Where L is the length of the pile, and $2L$ is the distance that the force wave will have to travel from the top of the pile to the toe of the pile (the boundary between the pile and soil) and then back up to the top of the pile.

The force at the top of the pile is measured indirectly by measuring the strain:

$$F(t) = \varepsilon(t) \cdot E \cdot A \tag{31}$$

Whereas velocity is calculated by integrating the acceleration (Likins & Rausche, 2014).

To separate the reflected force wave and the impulse force wave, the measurements are divided into upward and downward travelling components as follows (Likins & Rausche, 2014):

$$F_d = \frac{1}{2} \cdot [F(t) + Iv(t)] \tag{32}$$

$$F_u = \frac{1}{2} \cdot [F(t) - Iv(t)] \tag{33}$$

Where $I = \rho c A$; and while there is no reflection (and so no change in area or change in material) $Iv(t)$ will be equal to $F(t)$ and therefore the downward force will be the total force, and there will not be any upward force (as shown in figure 1).

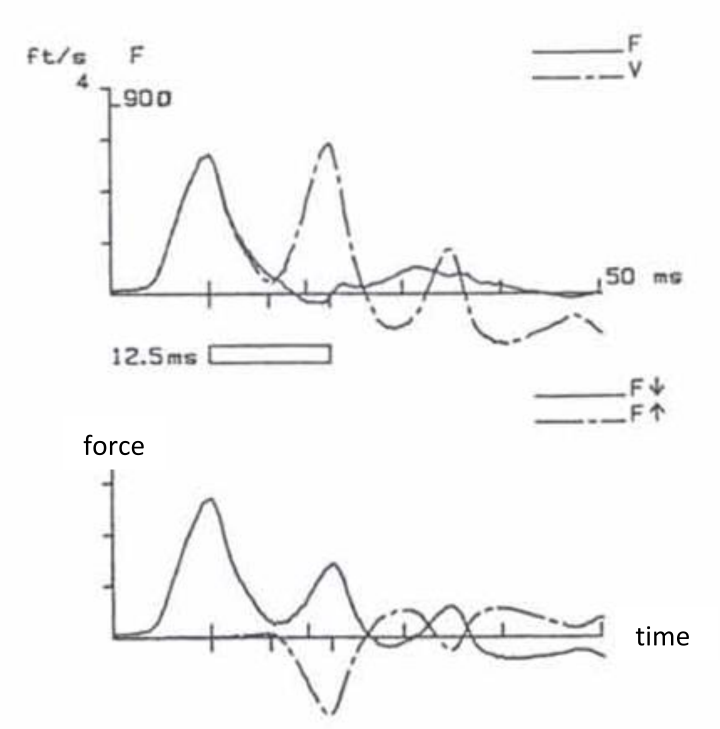

Figure 1 - Force measure at the pile top (Hussein & Rausche, 1991)

Figure 1 shows a real example of force measurement at the top of a pile. it is important to see that the upward force is only appearing at time $t = 2L/c$ while before that, the upward force is zero.

This statement is only true if the steel pile is uniform, on the contrary if any variable controlling the reflected force (ρ; c or A) should change, then a reflected force will be generated, and the latter will be recorded before $t = 2L/c$. Since ρ and c cannot change throughout the length of the pile, the only variable that is possible to have changed is the cross-section area which would indicate a damage on the pile (Hussein & Rausche, 1991).

The location of the damage can easily be found using the following equation:

$$x_{damage} = \frac{t_r \cdot c}{2} \quad (34)$$

3.3. Quantification of the Damage

The damage on a pile can be measured with many different parameters, but certainly one of the most important one is the cross-sectional area of the pile. Therefore, it would be possible to quantify the damage with the following formula (Rausche & Goble, 1988; Likins & Rausche, 2014; Hussein & Rausche, 1991):

$$\beta = \frac{A_2}{A_1} \quad (35)$$

Where β can go from 0 to 1, which indicates the percentage of the original cross-section that is still available. Therefore, lower numbers of beta, mean a higher degree of damage.

Recalling equation 29, and substituting $\rho c A$ with I, it is possible to state that:

$$F_r = \frac{(I_2 - I_1)}{I_2 + I_1} \cdot F_i \tag{36}$$

Where I_2 represents the properties of the damaged cross-section, and I_1 represents the original cross-section.

Knowing that the "upward" force, for the first reflection, is just the reflected force from the damages section of the pile, it is possible to state the following:

$$\frac{1}{2} \cdot [F(t) - Iv(t)] = \frac{(I_2 - I_1)}{I_2 + I_1} \cdot F_i \tag{37}$$

Considering that $F(t)$ and $Iv(t)$ are measurable quantities (as discussed in the beginning of the chapter), and setting a variable $\Delta = Iv(t) - F(t)$, the following equation can be derived:

$$\Delta = 2F_i \frac{(I_1 - I_2)}{I_2 + I_1} \tag{38}$$

It should be noted that the force that will impact the boundary at the damage, is not precisely F_i (so the force that the hammer impacted on the pile with), this will rather be reduced by a factor which could be defined as R, this should encapsulate damping of the soil for example, but also other variables, and therefore equation 38 would finally be (Rausche & Goble, 1988; Likins & Rausche, 2014):

$$\Delta = 2(F_i - R) \frac{(I_1 - I_2)}{I_2 + I_1} \tag{39}$$

Re-arranging the equation it is possible to discover that:

$$I_2 = \frac{I_1(2(F_i - R) - \Delta)}{2(F_i - R) + \Delta} \tag{40}$$

And so:

$$\frac{I_2}{I_1} = \frac{(F_i - R) - \Delta}{(F_i - R) + \Delta} \tag{41}$$

Considering that if the pile is made of the same material, c and ρ are constant:

$$\beta = \frac{(F_i - R) - \Delta}{(F_i - R) + \Delta} \tag{42}$$

Field tests have shown that, usually, the following empirical scale can be followed:

Table 1 - Beta and severity of damage (Rausche & Goble, 1988)

β	**Severity of damage**
1.0	Undamaged
0.8 – 1.0	Slightly damaged
0.6 – 0.8	Damaged
<0.6	Severely damaged

4. Conclusion and Final Considerations

The factor β is the main measure of damage to a pile due to hammering. This factor can be constantly calculated while hammering the pile, and as mentioned in the introduction of this report, if the start of a damage should be seen, the interruption of the hammering process could be considered, this is to preserve the integrity of the pile.

The same factor can be used to understand the left useful strength of the pile: if the strength of a pile is directly correlated to the area (as often is the case, especially for bearing piles), then the reduction in cross-sectional area can be used to estimate the reduction in useful bearing capacity.

Lastly, the wave equation analysis can also be useful to be used when estimating the bearing capacity of a pile itself. If no damage occurs prior to $t = 2L/c,$ then the reflected force is due to the resistance of the soil, and therefore the properties and bearing capacity of the soil can be estimated. This requires further research and deepening, and it is out of the scope of this report, but it is certainly something that could be of subject for future research.

5. References

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